JOURNAL OF APPROXIMATION THEORY 57, 278–292 (1989)

Rates for Approximation of Unbounded Functions by Positive Linear Operators

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Communicated by P. L. Butzer

Received November 12, 1986; revised November 17, 1987

DEDICATED TO PROFESSOR TADASHI KURODA ON HIS 60TH BIRTHDAY

Let $g_a(t)$ and $g_b(t)$ be two positive, strictly convex and continuously differentiable functions on an interval (a, b) $(-\infty \leqslant a < b \leqslant \infty)$, and let $\{L_n\}$ be a sequence of linear positive operators, each with domain containing 1, t, $g_a(t)$, and $g_b(t)$. If $L_n(f; x)$ converges to $f(x)$ uniformly on a compact subset of (a, b) for the test functions $f(t) = 1$, t, $g_a(t)$, $g_b(t)$, then so does every $f \in C(a, b)$ satisfying $f(t) = O(g_a(t))$ $(t \to a^+)$ and $f(t) = O(g_b(t))$ $(t \to b^-)$. We estimate the convergence rate of $L_n f$ in terms of the rates for the test functions and the moduli of continuity of f and f'. © 1989 Academic Press, Inc.

1. INTRODUCTION

The well-known Bohman–Korovkin theorem [1, 6] states that the (arbitrarily close uniform) approximation of all continuous real functions on $[a, b]$ by a sequence of positive linear operators is guaranteed by such an approximation for the three test functions: 1, t, and t^2. Various extension of this theorem to unbounded functions and quantitative results have been published in many papers. In particular, we cite works of Ditzian [3], Eisenberg and Wood [4], Ismail and May [6], Mamedov [8], Müller and Walk [10], Schurer [11], Sikkema [16], Swetits and Wood [17], and Walk [18]. In [12] the first author gives a version in which a strictly convex, continuously differentiable function $g(t)$ is used to replace t^2 as the third test function in order to approximate functions of the order of $g(t)$ as $|t| \to \infty$. It was subsequently applied in [12, 13, 14] to obtain many representation formulas for operator semigroups and cosine functions.

The purpose of this paper is to study the approximation of functions which may be unboundedly defined in a bounded set. Let g_a and g_b be two positive, strictly convex, and continuously differentiable functions on a

0021-9045/89 $3.00

finite or infinite open interval $I=(a, b)$ in R, and let $\{L_n\}$ be a sequence of positive linear operators on the space $C(I_1, g_a, g_b)$ of functions which are majorized near a and b by g_a and g_b, respectively, and are continuous on an interval $I_1 := [a_1, b_1] \subset I$. In Section 2, we shall establish a Bohman–Korovkin type approximation theorem for functions in $C(I_1, g_a, g_b)$ and a theorem presenting some quantitative estimates of the approximation.

Theorem 2.1 asserts that $L_n f$ is pointwise (uniformly) convergent to f on I_1 for all f in $C(I_1, g_a, g_b)$ if and only if it is so for the test functions: 1, t, $g_a(t)$, $g_b(t)$. In case $(a, b)=(-\infty, \infty)$ and $g_a = g_b$, this theorem is a special case of Theorem 2.2 of [12], in which functions on R^m have been treated. It is perhaps worth mentioning that, compared with other known results, the assumption: $L_n(t^2, x) \to x^2$ is not necessary in Theorem 2.1. For instance, if $L_n f \to f$ for $f(t)=1$, t, $|t|^{3/2}$, then by Theorem 2.1 it holds also for all f of the order of $|t|^{3/2}$ as $t \to \infty$, but, without the above assumption, theorems in [10, 18] could not be applied to.

Using the technique found in Shisha and Mond [15], DeVore [2], Ditzian [3], and Gonska [5], we obtain in Theorem 2.2 quantitative estimates for the convergence rate of $L_n f$ in terms of the rates for test functions 1, t, t^2, $g_s(t)$, $s=a, b$, and the moduli of continuity of f and f'.

In Section 3, applications to some well-known positive linear operators will be made. In some cases, slight modifications are made so that the operators can be applied to the desired functions.

2. The Approximation Theorems

Let $I=(a, b)$ $(-\infty \leqslant a < b \leqslant \infty)$ be the interval on which the functions to be approximated are defined, and let $I_1=[a_1, b_1]$ be contained in I. Let $g_a(t)$, $g_b(t)$ be given functions which are positive, strictly convex on I, continuously differentiable on I_1, and satisfy $g_b(t)=O(g_a(t))$ $(t \to a^+)$ and $g_a(t)=O(g_b(t))$ $(t \to b^-)$. If $a=-\infty$ [resp., $b=\infty$], we further assume that

$$\lim_{t \to -\infty} g_a(t)/|t| = \infty \qquad \left[\text{resp., } \lim_{t \to \infty} g_b(t)/|t| = \infty\right]. \tag{1}$$

For example, when $I=(0, \infty)$, $g_a(t)$ can be $(1+t)^{-1}$, t^p $(p>1)$, $t^{-\alpha}$, $\exp(wt^{-\alpha})$ $(\alpha, w>0)$, and $g_b(t)$ can be t^p, $\exp(wt^p)$ $(p>1, w>0)$. Obviously, $g_a(t)$ and $g_b(t)$ are continuous on I, and the functions

$$h_s(x, t)=g_s(t)-[g_s(x)+g'_s(x)(t-x)] \qquad (s=a, b) \tag{2}$$

are positive for $x \neq t$ and are continuous in (x, t) on $I_1 \times I$. The strict convexity of $g_s(t)$ also implies $h_s(x, t_1) < h_s(x, t_2)$ whenever $t_2 < t_1 \leqslant x$ or $t_2 > t_1 \geqslant x$ $(x \in I_1)$, and $h_s(x_1, t) < h_s(x_2, t)$ whenever $x_2 < x_1 \leqslant t$ or $x_2 > x_1 \geqslant t$ $(x_1, x_2 \in I_1)$. It follows that $h_s(x_1, t_1) \leqslant h_s(x_2, t_2)$ whenever $t_2 \leqslant t_1 \leqslant x_1 \leqslant x_2$ or $x_2 \leqslant x_1 \leqslant t_1 \leqslant t_2$ $(x_1, x_2 \in I_1)$. This property will be needed in the proofs of our theorems.

We shall denote by $C(I_1, g_a, g_b)$ the set of those functions $f(t)$ on I with the properties: (i) $f(t)$ is bounded on every compact subset of I; (ii) $f(t)$ is continuous at every point of I_1; (iii) $f(t) = O(g_a(t))$ $(t \to a^+)$ and $f(t) = O(g_b(t))$ $(t \to b^-)$. The functions 1, t, $g_a(t)$, and $g_b(t)$ are already contained in $C(I_1, g_a, g_b)$. For a positive linear operator L to operate on $C(I_1, g_a, g_b)$, we require that $L(g_a; x) < \infty$ and $L(g_b; x) < \infty$ for all $x \in I_1$.

For positive linear operators L_n on $C(I_1, g_a, b_b)$, $\alpha_n^2(x)$ and $\beta_n^2(x)$ will denote the functions $L_n(h_a(x, t); x)$ and $L_n(h_b(x, t); x)$, respectively, and $\gamma_n^2(x)$ will denote $L_n((t-x)^2; x)$ whenever it is defined. If we write $L_n(t^i; x)$ $x^i + \lambda_{ni}(x)$ $(i = 0, 1, 2)$ and $L_n(g_s; x) = g_s(x) + \mu_{ns}(x)$ $(s = a, b)$, then

$$\alpha_n^2(x) = \mu_{na}(x) - g_a(x)\,\lambda_{n0}(x) - g_a'(x)\,\lambda_{n1}(x) + g_a'(x)\,x\lambda_{n0}(x), \tag{3}$$

$$\beta_n^2(x) = \mu_{nb}(x) - g_b(x)\,\lambda_{n0}(x) - g_b'(x)\,\lambda_{n1}(x) + g_b'(x)\,x\lambda_{n0}(x), \tag{4}$$

$$\gamma_n^2(x) = \lambda_{n2}(x) - 2x\lambda_{n1}(x) + x^2\lambda_{n0}(x). \tag{5}$$

For a fixed $x \in I_1$, $\alpha_n^2(x) \to 0$ and $\beta_n^2(x) \to 0$ when $L_n(t^i; x) \to x^i$ $(i = 0, 1)$ and $L_n(g_s; x) \to g_s(x)$ $(s = a, b)$; $\gamma_n^2(x) \to 0$ when $L_n(t^i; x) \to x^i$ $(i = 0, 1, 2)$. The assertions hold for uniform convergence on I_1 too.

THEOREM 2.1. *Let $\{L_n\}$ be a sequence of positive linear operators on $C(I_1, g_a, g_b)$. If $L_n(t^i; x) \to x^i$ $(i = 0, 1)$ and $L_n(g_s; x) \to g_s(x)$ $(s = a, b)$ for some $x \in I_1$ (resp. uniformly for all $x \in I_1$), then for any f in $C(I_1, g_a, g_b)$, $L_n(f; x)$ converges to $f(x)$ at x (resp. uniformly on I_1).*

LEMMA. *If $f \in C(I_1, g_a, g_b)$ and if $0 < \delta < \min\{a_1 - a, b - b_1\}$, then there is a constant $M(f, I_1, \delta)$ such that*

$$|f(t) - f(x)| \leqslant M(f, I_1, \delta)[h_a(x, t) + h_b(x, t)] \tag{6}$$

for all $x \in I_1$ and $t \in I$ with $|t - x| \geqslant \delta$.

Proof. First, we suppose $\lim_{t \to a^+} g_a(t) = \infty$. Since f, g_a, and g_a' are continuous on the compact set I_1, they are bounded there by a number $k > 0$. Hence we have

$$\frac{|f(t) - f(x)|}{h_a(x, t)} \leqslant \frac{|f(t)| + k}{g_a(t)} \left\{1 - \frac{k}{g_a(t)} - k\,\frac{|t| + k}{g_a(t)}\right\}^{-1}$$

for all $x \in I_1$ and for t sufficiently close to a. Now the assumptions:

$f(t)=O(g_a(t))$ $(t \to a^+)$ and (1) (in case $a=-\infty$) imply the existence of a $M_1>0$ and a point $a' \in (a, a_1-\delta)$ such that

$$|f(t)-f(x)| \leqslant M_1 h_a(x, t) \qquad (x \in I_1, t \in (a, a']). \tag{7}$$

Next, suppose $\lim_{t \to a^+} g_a(t) < \infty$. Then $f(t)$ is bounded on $(a, b_1]$, and (7) still holds with $M_1 = 2 \sup\{|f(t)|;\ t \in (a, b_1]\}/h_a(a_1, a_1-\delta)$. Here we note that for $x \in I_1$ and $t \in (a, a']$, $h_a(x, t) \geqslant h_a(a_1, a') > h_a(a_1, a_1-\delta) > 0$ as remarked before.

Similarly, there exist $M_2>0$ and $b' \in (b_1+\delta, b)$ such that

$$|f(t)-f(x)| \leqslant M_2 h_b(x, t) \qquad (x \in I_1, t \in b', b)). \tag{8}$$

Finally, since $h_a(x, t)$ is positive and continuous on the compact set $\{(x, t); x \in I_1, t \in [a', b'], |x-t| \geqslant \delta\}$, it assumes a positive minimum m. Hence we have for (x, t) in this set,

$$|f(t)-f(x)| \leqslant (2/m) \sup\{|f(t)|; t \in [a', b']\}\ h_a(x, t) = M_3 h_a(x, t). \tag{9}$$

Combining (7), (8), and (9), we obtain (6) with $M(f, I_1, \delta) = M_1 + M_2 + M_3$.

Proof of Theorem 2.1. It follows from the lemma that

$$|f(t)-f(x)| \leqslant \omega(f, I_1, \delta) + M(f, I_1, \delta)[h_a(x, t) + h_b(x, t)]$$

for all $x \in I_1$ and $t \in I$, where $\omega(f, I_1, \delta) = \sup\{|f(x)-f(t)|;\ x \in I_1,\ t \in I, |x-t| \leqslant \delta\}$. $\omega(f, I_1, \delta)$ tends to 0 with δ because f is continuous at every point of I_1. Hence, on applying L_n, we have

$$\begin{aligned} |L_n(f; x)-f(x)| \leqslant{} & |f(x)|\ |L_n(1; x)-1| + \omega(f, I_1, \delta)\, L_n(1; x) \\ & + M(f, I_1, \delta)[\alpha_n^2(x) + \beta_n^2(x)], \end{aligned}$$

and the proof is complete.

With regard to the convergence rate, we give some estimates in the following

THEOREM 2.2. *Let c be a positive number. Let $I_2 = [a_2, b_2]$ and $\eta > 0$ be such that $[a_2-\eta, b_2+\eta] \subset I_1 = [a_1, b_1]$, and let $L_n(t^2; x)$ be defined.*

If f belongs to $C(I_1, g_a, g_b)$, then there exists a constant $K(f, I_2, \eta)$ such that for every $x \in I_2$

$$\begin{aligned} |L_n(f; x)-f(x)| \leqslant{} & |f(x)|\ |L_n(1; x)-1| + \omega(f, I_1, c\gamma_n(x))[L_n(1; x) + c^{-2}] \\ & + K(f, I_2, \eta)[\alpha_n^2(x) + \beta_n^2(x)]. \end{aligned} \tag{10}$$

If, in addition, f' is continuous on I_1, then

$$|L_n(f;x)-f(x)| \leqslant |f(x)|\ |L_n(1;x)-1) + K(f, I_2, \eta)(\alpha_n^2(x)+\beta_n^2(x)] + |f'(x)|\, L_n(|t-x|;x) + \omega(f', I_1, c\gamma_n(x))[L_n(|t-x|;x) + (2c)^{-1}\gamma_n(x)] \quad (11)$$

$$\leqslant |f(x)|\ |L_n(1;x)-1| + K(f, I_2, \eta)[\alpha_n^2(x)+\beta_n^2(x)] + \gamma_n(x)\{|f'(x)|(L_n(1;x))^{1/2} + \omega(f', I_1, c\gamma_n(x))[(L_n(1;x))^{1/2} + (2c)^{-1}]\} \quad (12)$$

and

$$|L_n(f;x)-f(x)| \leqslant |f(x)|\ |L_n(1;x)-1) + K(f, I_2, \eta)[\alpha_n^2(x)+\beta_n^2(x)] + \|f'\|_{I_1} L_n(|t-x|;x) + \omega(f', I_1, c)[L_n(|t-x|;x) + (2c)^{-1}\gamma_n^2(x)] \quad (13)$$

$$\leqslant |f(x)|\ |L_n(1;x)-1| + K(f, I_2, \eta)[\alpha_n^2(x)+\beta_n^2(x)] + \gamma_n(x)\{\|f'\|_{I_1}(L_n(1;x))^{1/2} + \omega(f', I_1, c)[(L_n(1;x))^{1/2} + (2c)^{-1}\gamma_n(x)]\}. \quad (14)$$

Remark. If we substitute the c in (13) and (14) by $c\gamma_n(x)$, then these two estimates will become (11) and (12) with $|f'(x)|$ replaced by $\|f'\|$. We shall use different techniques to derive these two pairs of estimates.

Proof. First, we let $f(t)$, I_2, and η play the respective roles of $f(t)-f(x)$, I_1, and δ in the previous lemma. Then the same argument will provide two constants K_1', K_2', and two points $a' \in (a, a_2-\eta)$ and $b' \in (b_2+\eta, b)$ such that

$$|f(t)| \leqslant K_1' h_a(x,t) \qquad (x \in I_2,\ t \in (a, a']),$$

$$|f(t)| \leqslant K_2' h_b(x,t) \qquad (x \in I_2,\ t \in [b', b)).$$

Next, for $x \in I_2$, $t \in [a', a_2-\eta]$, we have

$$|f(t)| \leqslant \sup\{|f(t)|;\ t \in [a', a_2-\eta]\}(h_a(a_2, a_2-\eta))^{-1} h_a(x,t) = K_1'' h_a(x,t)$$

(recall the remark on $h_a(x,t)$). It follows that for $x \in I_2$ and $t \in (a, a_2-\eta]$ we have

$$|f(t)| \leqslant \max\{K_1', K_1''\}\, h_a(x,t)$$

and

$$|f(x)| \leqslant \|f\|_{I_2} h_a(x,t)/h_a(a_2, a_2-\eta) = K_1 h_a(x,t).$$

Similarly, the following two estimates hold for $x \in I_2$, $t \in [b_2+\eta, b)$:

$$|f(t)| \leqslant \max\{K_2', K_2''\}\, h_b(x, t),$$

$$|f(x)| \leqslant \|f\|_{I_2}\, h_b(x, t)/h_b(b_2, b_2+\eta) = K_2 h_b(x, t).$$

Combining these estimates we have for $x \in I_2$ and $t \in I - [a_2-\eta, b_2+\eta]$,

$$|f(t)-f(x)| \leqslant |f(t)| + |f(x)| \leqslant K(f, I_2, \eta)[h_a(x, t) + h_b(x, t)], \quad (15)$$

where $K(f, I_2, \eta) = \max\{K_1, K_2\} + \max\{K_1', K_1'', K_2', K_2''\}$.

For fixed $x \in I_2$, any $t \in [a_2-\eta, b_2+\eta]$, and $\delta > 0$ we have, as in [15] (with $\omega(\cdot) = \omega(f, I_1, \cdot)$),

$$|f(t)-f(x)| \leqslant \omega(|t-x|) \leqslant (1+(t-x)^2\,\delta^{-2}\,\omega(\delta). \quad (16)$$

If, in addition, $f' \in C(I_1)$, then, as in [19],

$$\begin{aligned} |f(t)-f(x)| &\leqslant |f'(x)(t-x)| + \left|\int_x^t |f'(u)-f'(x)|\, du\right| \\ &\leqslant |f'(x)|\;|t-x| + \left|\int_x^t (1+|u-x|\,\delta^{-1})\,\omega(f', I_1, \delta)\, du\right|. \quad (17)\end{aligned}$$

Combining (15) and (16) we can deduce for all t in I,

$$|f(t)-f(x)| \leqslant (1+(t-x)^2\delta^{-2})\,\omega(\delta) + K(f, I_2, \eta)[h_a(x, t) + h_b(x, t)],$$

from which (10) follows easily, by applying L_n and then letting $\delta \downarrow c\gamma_n(x)$ (c.f. Mond [9]). The estimate in (11) and (12) is derived in the same way, using (17) instead of (16) and using the fact that $L_n(|t-x|; x) \leqslant \gamma_n(x)^{1/2}(L_n(1; x))^{1/2}$.

For the proof of (13) and (14) we shall use the technique in [5]. Let f be in $C(I_1, g_a, g_b)$ and continuously differentiable on I_1. We define

$$\hat{f}'(x) = \begin{cases} f'(a) & \text{for } x < a_1, \\ f'(x) & \text{for } a_1 \leqslant x \leqslant b_1, \\ f'(b) & \text{for } x > b_1, \end{cases}$$

and

$$k(x) = (2c)^{-1} \int_{-c}^{c} \hat{f}'(x+s)\, ds \qquad \text{for } x \in I_1 = [a_1, b_1].$$

Then $k \in C^1(I_1)$ and we have for $x \in I_1$ that $|k(x)| \leqslant \|f'\|_{I_1}$ and

$$\begin{aligned} |k'(x)| &= |(2c)^{-1}[\hat{f}'(x+c) - \hat{f}'(x-c)]| \\ &\leqslant (2c)^{-1}\,[|\hat{f}'(x+c) - f'(x)| + |f'(x) - \hat{f}'(x-c)|] \\ &\leqslant c^{-1}\omega(f', I_1, c). \end{aligned}$$

There exists a $\theta_x \in [-c, c]$ such that $k(x) = \hat{f}'(x+\theta_x)$ and so

$$|(f'-k)(x)| = |f'(x) - \hat{f}'(x+\theta_x)| \leqslant \omega(f', I_1, c).$$

Thus, if we choose a g in $C^2(I_1)$ such that $g' = k$, then

$$\|g'\|_{I_1} \leqslant \|f'\|_{I_1}, \; \|g''\|_{I_1} \leqslant c^{-1}\omega(f', I_1, c)$$

and

$$\|(f-g)'\|_{I_1} \leqslant \omega(f', I_1, c).$$

Now for $x \in I_2$ and $t \in [a_2 - \eta, b_2 + \eta]$,

$$\begin{aligned} |f(t) - f(x)| &\leqslant |(f-g)(t) - (f-g)(x)| + |g(t) - g(x)| \\ &= |(f-g)'(u)(t-x)| + \left| g'(x)(t-x) + \frac{g''(v)}{2}(t-x)^2 \right| \\ &\leqslant [\|(f-g)'\| + \|g'\|]|t-x| + \frac{1}{2}\|g''\|(t-x)^2 \\ &\leqslant [\omega(f', I_1, c) + \|f'\|]\,|t-x| + \frac{1}{2c}\omega(f', I_1, c)(t-x)^2. \end{aligned}$$

Combining this and (15) we obtain that for $x \in I_2$ and $t \in I$

$$\begin{aligned} |f(t) - f(x)| \leqslant K(f, I_2, \eta)[h_a(x,t) + h_b(x,t)] + [\omega(f', I_1, c) + \|f'\|]|t-x| \\ + \frac{1}{2c}\omega(f', I_1, c)(t-x)^2, \end{aligned}$$

from which (13) and (14) follow immediately by applying L_n.

Remark. Under the assumption that $L_n f \to f$ for $f(t) = 1, t, t^2$, Walk [18] and Müller and Walk [10] have considered the approximation of a function f which satisfies $\sup L_n(|f|^p; x) < \infty$, $x \in (a, b)$ for some $p > 1$. One might expect to derive Theorems 2.1 and 2.2 from their theorems. This turns out to be not possible. Even if one assumes that $L_n f \to f$ for $f(t) = t^2$, $t \in R$, in addition to 1, t, and $g(t)$, in order to use the theorems of [10, 18] to assert that $L_n f \to f$ for a function f in $C(I_l, g, g)$ (as one can use Theorem 2.1 to do so), according to [18, Remark 1(b)], one has to find a $p > 1$ such that

$$|f(t)|^p \leqslant g(t) \qquad (t \in R).$$

But this is not always possible. For instance, if $g(t) = \exp(t^2 + |t|)$ and

$f(t)=\exp(t^2)$, then $f(t)=O(g(t))$ $(|t|\to\infty)$ (that is $f\in C(I_1, g, g)$) but there is no $p>1$ such that

$$|f(t)|^p \leqslant g(t)$$

for $t\in R$.

3. Examples

In this section we shall modify some well-known linear positive operators so as to approximate unbounded functions on, e.g., $(0, 1)$ or $(0, \infty)$. The results in Section 2 will be applied to yield some estimates of convergence rate for these operators.

EXAMPLE 1. Let $I=(0,1)$ and $I_1=[a_1, b_1]\subset I$. The operators B_n: $C(I_1, 1/t, 1/(1-t))\to C(I_1)$ defined by

$$B_n(f(t); x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k+1}{n+2}\right)$$

are the Bernstein operators with $f(k/n)$ replaced by $f((k+1)/(n+2))$. This modification enables B_n to operate on functions which are unbounded near 0 and 1. Straightforward calculations give

$$B_n(1; x)=1, \qquad B_n(t; x)=x+\frac{1-2x}{n+2},$$

$$B_n(t^2; x)=x^2+[-(5n+4)\,x^2+3nx+1](n+2)^{-2},$$

$$B_n\left(\frac{1}{t}; x\right)=\frac{1}{x}+\frac{1}{n+1}\frac{1}{x}-\frac{n+2}{n+1}(1-x)^{n+1}\frac{1}{x},$$

$$B_n\left(\frac{1}{1-t}; x\right)=\frac{1}{1-x}+\frac{1}{n+1}\frac{1}{1-x}-\frac{n+2}{n+1}\frac{1}{1-x}x^{n+1}.$$

Hence, by (3), (4), and (5), we have for $x\in I_1$

$$\alpha_n^2(x)=\frac{1}{n+1}\frac{1}{x}-\frac{n+2}{n+1}\frac{1}{x}(1-x)^{n+1}+x^{-2}\frac{1-2x}{n+2}$$

$$\leqslant\frac{1}{n+2}(x^{-2}+x^{-1})-x^{-1}(1-x)^{n+1}\leqslant\frac{2}{n}/a_1^2-(1-b_1)^{n+1}/b_1,$$

$$\beta_n^2(x)\leqslant\frac{1}{n+2}[(1-x)^{-2}+(1-x)^{-1}]-x^{n+1}(1-x)^{-1}$$

$$\leqslant 2(1-b_1)^{-2}/n-a_1^{n+1}/(1-a_1)$$

and

$$\gamma_n^2(x) = [-(5n+4)\,x^2 + 3nx + 1]/(n+2)^2 - 2x\,\frac{1-2x}{n+2} \leqslant \frac{1}{4n}.$$

Therefore, for $f \in C(I_1, 1/t, 1/(1-t))$ we have

$$\|B_n(f(t); x) - f(x)\|_{I_1} \to 0 \qquad \text{as } n \to \infty,$$

$$|B_n(f; x) - f(x)| \leqslant (1 + c^{-2})\,\omega\left(f, I_1, c\left(\frac{1}{4n}\right)^{1/2}\right) + K_1(f, a_1, b_1, \eta)n^{-1},$$

$$|B_n(f; x) - f(x)| \leqslant \left\{|f'(x)| + \left(1 + \frac{1}{2c}\right)\omega\left(f', I_1, c\left(\frac{1}{4n}\right)^{1/2}\right)\right\}\left(\frac{1}{4n}\right)^{1/2} + K_1(f, a_1, b_1, \eta)\,n^{-1},$$

and

$$|B_n(f; x) - f(x)| \leqslant \left\{\|f'\|_{I_1} + \omega(f', I_1, c)\left[1 + \frac{1}{2c}\left(\frac{1}{4n}\right)^{1/2}\right]\right\}\left(\frac{1}{4n}\right)^{1/2} + K_1(f, a_1, b_1, \eta)\,n^{-1}$$

for $x \in I_2 = [a_1 + \eta, b_1 - \eta]$, $\eta > 0$.

EXAMPLE 2. The operators $M_n\colon C(I_1, 1/t, 1/1-t) \to C(I_1)$ defined by

$$M_n(f(t); x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f\left(\frac{k+1}{n+k+1}\right)$$

are the Meyer–König and Zeller operators with $f(k/(n+k))$ replaced by $f((k+1)/(n+k+1))$. We have $M_n(t^i; x) = x^i + \lambda_{ni}(x)$ $(i = 0, 1, 2)$ with $\lambda_{n0}(x) = 0$,

$$0 < \lambda_{n1}(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k \left(\frac{k+1}{n+k+1} - \frac{k}{n+k}\right)$$

$$\leqslant (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k \frac{1}{n} = \frac{1}{n},$$

$$0 < \lambda_{n2}(x) = (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k \times \left[\left(\frac{k+1}{n+k+1}\right)^2 - \frac{k}{n+k}\,\frac{k-1}{n+k-1}\right] < \frac{3}{n},$$

$$M_n\left(\frac{1}{t}; x\right) = \frac{1}{x}(1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k+1}{k+1} x^{k+1}$$
$$= \frac{1}{x}(1-(1-x)^{n+1}) = \frac{1}{x} - (1-x)^{n+1} x^{-1},$$

and

$$M_n\left(\frac{1}{1-t}; x\right) = \frac{n+1}{n}(1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k+1}{k} x^k$$
$$= \frac{n+1}{n}\frac{1}{1-x} = \frac{1}{1-x} + \frac{1}{n(1-x)}.$$

It follows from (3), (4), and (5) that for $x \in I_1$

$$\alpha_n^2(x) \leqslant -(1-x)^{n+1} x^{-1} + n^{-1}x^{-2} \leqslant a_1^{-2} n^{-1} - (1-b_1)^{n+1}/b_1,$$
$$\beta_n^2(x) = 1/(1-x)\, n - \lambda_{n1}(x)(1-x)^{-2} \leqslant (2-x)/(1-x)^2 n$$
$$\leqslant 2(1-b_1)^{-2} n^{-1},$$
$$\gamma_n^2(x) = \lambda_{n2}(x) - 2x\, \lambda_{n1}(x) < \frac{3}{n} + \frac{2x}{n} < \frac{5}{n}.$$

Therefore, for f in $C(I_1, 1/t, 1/(1-t))$ we have

$$\|M_n(f(t); x) - f(x)\|_{I_1} \to 0 \qquad \text{as } n \to \infty,$$
$$|M_n(f; x) - f(x)| \leqslant (1 + c^{-2})\, \omega(f, I_1, c(5/n)^{1/2}) + K_1(f, a_1, b_1, \eta)\, n^{-1},$$
$$|M_n(f; x) - f(x)| \leqslant \left\{ |f'(x)| + \left(1 + \frac{1}{2c}\right) \omega(f', I_1, c(5/n)^{1/2}) \right\} (5/n)^{1/2}$$
$$+ K_1(f, a_1, b_1, \eta)\, n^{-1},$$

and

$$|M_n(f; x) - f(x)| \leqslant \{ \|f'\|_{I_1} + (1 + (5/n)^{1/2} (2c)^{-1}\, \omega(f', I_1, c) \} (5/n)^{1/2}$$
$$+ K_1(f, a_1, b_1, \eta)\, n^{-1}$$

for $x \in I_2 = [a_1 + \eta, b_1 - \eta]$, $\eta > 0$.

EXAMPLE 3. Let $I = (0, \infty)$, $I_1 = [a_1, b_1] \subset I$. We consider the operator $B_n \colon C(I_1, 1/t, e^{wt}) \to C(I_1)$ defined by

$$B_n(f(t); x) = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k (1+x)^{-n-k} f\left(\frac{k+1}{n}\right).$$

These are the special Baskakov operators with $f(k/n)$ replaced by $f((k+1)/n)$. For these operators we have

$$B_n(1;x)=1, \qquad B_n(t;x)=x+\frac{1}{n},$$

$$B_n(t^2;x)=x^2+[x^2/n+3x/n+1/n^2],$$

$$B_n\left(\frac{1}{t};x\right)=\frac{n}{n-1}\frac{1}{x}[B_{n-1}(1;x)-(1+x)^{-n+1}]$$

$$=\frac{1}{x}+\frac{1}{n-1}\frac{1}{x}-\frac{n}{n-1}\frac{1}{x}(1+x)^{-n+1}, \qquad n\geqslant 2,$$

$$B_n(e^{wt};x)=e^{w/n}\sum_{k=0}^{\infty}\binom{-n}{k}(-xe^{w/n})^k(1+x)^{-n-k}$$

$$=e^{w/n}[1+x-xe^{w/n}]^{-n}=e^{wx}+\mu_n(x),$$

where $\mu_n(x)=e^{w/n}[1+x-xe^{w/n}]^{-n}-e^{wx}$ converges to 0 uniformly on $[0,\theta]$ for any $\theta>0$ (see [12, Theorem 3.6]). Now substitutions into (3), (4), and (5) yield

$$\alpha_n^2(x)=\frac{1}{n-1}\frac{1}{x}-\frac{n}{n-1}\frac{1}{x}(1+x)^{-n+1}+x^{-2}\frac{1}{n},$$

$$\beta_n^2(x)=e^{w/n}[1+x-x\,e^{w/n}]^{-n}-e^{wx}-we^{wx}\frac{1}{n},$$

$$\gamma_n^2(x)=\frac{x^2}{n}+\frac{3x}{n}+1/n^2-\frac{2x}{n}=\frac{x^2}{n}+\frac{x}{n}+n^{-2}\leqslant(b_1+1)^2/n$$

for $x\in I_1$. It is clear that these three sequences converge to 0 uniformly for x in I_1. Hence Theorems 2.1 and 2.2 imply that if f belongs to $C(I_1, 1/t, e^{wt})$, then we have

$$\|B_n(f(t);x)-f(x)\|_{I_1}\to 0 \qquad \text{as} \quad n\to\infty;$$

$$|B_n(f;x)-f(x)|\leqslant(1+c^{-2})\,\omega(f,I_1,c(b_1+1)\,n^{-1/2})$$
$$+K(f,I_2,\eta)(\alpha_n^2(x)+\beta_n^2(x)),$$

$$|B_n(f;x)-f(x)|\leqslant\{|f'(x)|+\left(1+\frac{1}{2c}\right)\omega(f',I_1,c(b_1+1)\,n^{-1/2}\}$$
$$\times(b_1+1)\,n^{-1/2}+K(f,I_2,\eta)(\alpha_n^2(x)+\beta_n^2(x)),$$

and

$$|B_n(f; x)-f(x)| \leqslant \left\{\|f'\|_{I_1}+\left[1+\frac{1}{2c}(b_1+1)\,n^{-1/2}\right]\omega(f', I_1, c)\right\}$$
$$\times (b_1+1)\,n^{-1/2}+K(f, I_2, \eta)(\alpha_n^2(x)+\beta_n^2(x))$$

for $x \in I_2=[a_1+\eta, b_1-\eta]$, $\eta>0$.

EXAMPLE 4. Let $I=(0, \infty)$ and $I_1=[a_1, b_1]$, and let $S_n\colon C(I_1, 1/t, e^{wt}) \to C(I_1)$ be defined by

$$S_n(f(t); x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k+1}{n}\right).$$

These are the Mirakjan–Szász operators with $f(k/n)$ replaced by $f((k+1)/n)$. We have $S_n(1; x)=1$, $S_n(t; x)=x+n^{-1}$, $S_n(t^2; x)=x^2+3x/n+n^{-2}$, $S_n(1/t; x)=(1/x)\,e^{-nx}\sum_{k=0}^{\infty}((nx)^{k+1}/(k+1)!)=1/x-(1/x)\,e^{-nx}$, and $S_n(e^{wt}; x)=\exp[nx(e^{w/n}-1)]\,e^{w/n}=e^{wx}+\mu_n(x)$, where $\mu_n(x)$ converges to 0 uniformly for x in I_1 (cf. [6]). It follows that for f in $C(I_1, 1/t, e^{wt})$

$$\|S_n(f(t); x)-f(x)\|_{I_1} \to 0 \qquad \text{as} \quad n \to \infty.$$

Moreover, (10), (11), (12), (13), and (14) will hold with

$$\alpha_n^2(x)= -\frac{1}{x}e^{-nx}+x^{-2}n^{-1} \leqslant a_1^{-2}\,n^{-1}-b_1^{-1}\,e^{-nb_1},$$
$$\beta_n^2(x)=\exp[nx(e^{w/n}-1)+w/n]-e^{wx}-w\,e^{wx}\,n^{-1},$$
$$\gamma_n^2(x)=3xn^{-1}+n^{-2}-2xn^{-1}<(b_1+1)\,n^{-1}.$$

EXAMPLE 5. For $I_1=[a_1, b_1] \subset I=(0, \infty)$ and for any $k=1, 2, \ldots$, the Post–Widder operators $P_n\colon C(I_1, t^{-k}, e^{wt}) \to C(I_1)$ are defined by

$$P_n(f(t); x)=\frac{(n/x)^n}{(n-1)!}\int_0^{\infty} e^{-nt/x}\,t^{n-1}\,f(t)\,dt.$$

On substituting $s=n/x$ into the identity

$$\int_0^{\infty} e^{-st}\,t^{n+i-1}\,e^{wt}\,dt=(n+i-1)!\,(s-w)^{-n-i} \qquad (n+i \geqslant 1, s>w),$$

we derive that $P_n(t^i\,e^{wt}; x)=[(n+i-1)!/(n-1)!](n/x)^n(n/x-w)^{-n-i}$

holds when $n+i \geqslant 1$ and $n/x > w$. Thus, taking suitable values of i and w we obtain the following identities:

$$P_n(1; x) = 1, \qquad P_n(t; x) = x, \qquad P_n(t^2; x) = x^2 + x^2/n,$$

$$P_n(t^{-k}; x) = x^{-k} + x^{-k}\left[\frac{n^k}{(n-1)(n-2)\cdots(n-k)} - 1\right], \qquad n \geqslant k+1,$$

and

$$P_n(e^{wt}; x) = (1 - wx/n)^{-n} = e^{wx} + (1 - wx/n)^{-n} - e^{wx}.$$

Since the last two sequences converge uniformly on I_1 to x^{-k} and e^{wx}, respectively, Theorem 2.1 implies

$$\|P_n(f(t); x) - f(x)\|_{I_1} \to 0$$

for all f in $C(I_1, t^{-k}, e^{wt})$ $(k, w > 0)$. Moreover, (10), (11), (12), (13), and (14) will hold with

$$\alpha_n^2(x) = x^{-k}\left[\frac{n^k}{(n-1)(n-2)\cdots(n-k)} - 1\right],$$

$$\beta_n^2(x) = \left(1 - \frac{wx}{n}\right)^{-n} - e^{wx},$$

$$\gamma_n^2(x) = \frac{x^2}{n} \leqslant \frac{b_1^2}{n}.$$

EXAMPLE 6. For $I_1 = [a_1, b_1] \subset I = (0, \infty)$ and for $w > 0$, $k = 1, 2, \ldots$, the Gamma operators G_n: $C(I_1, e^{w/t}, t^k) \to C(I_1)$ are defined by

$$G_n(f(t); x) = \frac{x^{n+1}}{n!} \int_0^\infty e^{-xt}\, t^n f\left(\frac{n+1}{t}\right) dt.$$

It is known that $G_n 1 = 1$, $G_n t = x + x/n$, and $G_n t^2 = x^2 + ((3n+1)/n(n-1))\, x^2$. Also we have

$$G_n(e^{w/t}; x) = \frac{x^{n+1}}{n!} \int_0^\infty \exp\left[-\left(x - \frac{w}{n+1}\right) t\right] t^n\, dt$$

$$= x^{n+1}\left(x - \frac{w}{n+1}\right)^{-n-1} = \left(1 - \frac{w}{(n+1)\,x}\right)^{-n-1}$$

and

$$G_n(t^k; x) = x^k\, \frac{(n+1)^k}{n(n-1)\cdots(n-k+1)}.$$

It can easily be shown that $G_n t$, $G_n e^{w/t}$, and $G_n t^k$ converge uniformly on I_1 to x, $e^{w/x}$, and x^k, respectively. Hence we can deduce from Theorem 2.1 that for all f in $(I_1, e^{w/t}, t^k)$

$$\|G_n(f(t); x) - f(x)\|_{L_1} \to 0 \qquad \text{as} \quad n \to \infty.$$

In addition, (10), (11), (12), (13), and (14) will hold with

$$\alpha_n^2(x) = -e^{w/t} + \left(1 - \frac{w}{(n+1)x}\right)^{-n-1} + \frac{w}{nx} e^{w/x},$$

$$\beta_n^2(x) = x^k \left[\frac{(n+1)^k}{n(n-1)\cdots(n-k+1)} - 1\right] - kn^{-1}x^k,$$

$$\gamma_n^2(x) = \frac{3n+1}{n(n-1)} x^2 - \frac{2x^2}{n}.$$

Acknowledgments

The authors thank the referees for their helpful suggestions.

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